Subdisions of shellable complexes

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Background

Let Δ be a $(d-1)$-dimensional simplicial complex. We have the following polynomials:

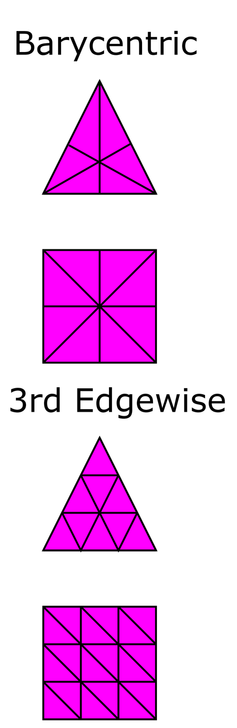
f-poynomial: $f(\Delta; x) = f_{-1} + f_0 x + \cdots + f_{d-1}x^d$ where $f_{-1} = 1$ and f_i is the number of i-dimensonal faces of Δ.

h-polynomial: $h(\Delta; x) = (1-x)^d f\left(\frac{x}{1-x}\right)$

A well-studied question is the classification of h-polynomials of simplicial complexes. For example, we may ask whether a h-polynomial is

- **unimodal:** $h_0 \leq h_1 \leq \cdots \leq h_k \geq \cdots \geq h_d$
- **real-rooted:** stronger than unimodality

Theorem (Brenti, Welker 2008) [2] If $h(\Delta; x)$ has positive coefficients,the barycentric subdivision of Δ has a real-rooted h-polynomial.



Question: What other subdivisions of polytopal complexes have a real-rooted h-polynomial? Some places to start:

1. **barycentric subdivision** of boundary complexes of cubical polytopes (Brenti, Welker)
2. **edgewise subdivision** of boundary complexes of cubical and simplicial polytopes (Mohammadi, Welker)

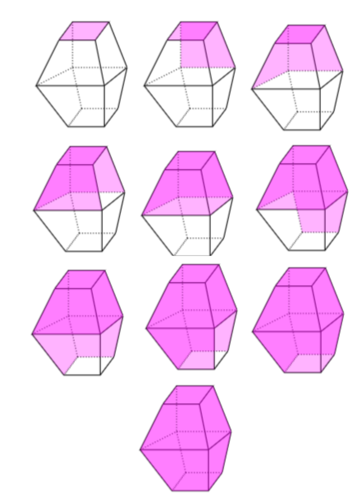
Sketch of technique

1. Use the idea of shelling to decompose our subdivision into disjoint pieces.
2. Find all possible h- polynomials of these bite-sized pieces.
3. Use the idea of interlacing polynomials to show that the $h-$ polynomial of the whole subdivision is real-rooted.

Shelling and relative complexes

Shelling— A linear ordering $(F_1, \ldots, F_s)$ on the facets of a pure polytopal complex such that:

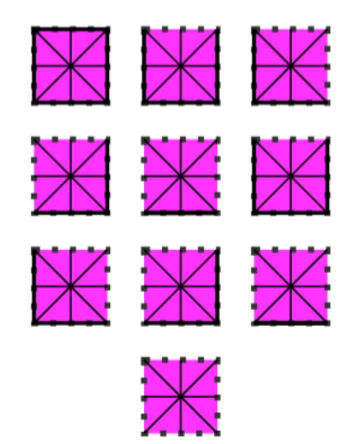
1. Boundary complex of F_1 is shellable.
2. The intersection of F_j with the prev. facets is nonempty and is the beginning segment of a shelling of the boundary complex of F_j.



Relative complex– $\mathcal{C}/\mathcal{D}$ denotes a polytopal complex $\mathcal{C}$ with a subcomplex D removed.

For subdivision $\mathcal{C}'$ of a shellable complex $\mathcal{C}$, we use our shelling order to define:

$$\mathcal{R}_i = \mathcal{C}'|_{F_i} / \left(\bigcup_{k=1}^{i-1} \mathcal{C}'|_{F_k}\right)$$



Interlacing polynomials

Interlacing polynomials: there is a zero of p between each pair of zeroes of q and vice versa for real-rooted polynomials p and q.

$\mathbf{p} \prec \mathbf{q}$: p and q are interlacing and $p'q - q'p \geq 0$

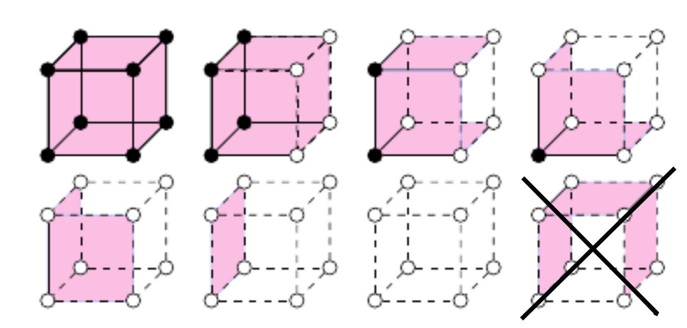
Theorem (Borcea, Brändén) Suppose $p_1 \prec p_2 \prec \cdots \prec p_n$. Then any nonnegative combination of $p_1 \ldots p_n$ is real-rooted.

Stable shellings

A shelling is **stable** if the relative complexes are stable (satisfying a specific property related to face lattices).

Simpicial complexes: all shellings are stable.

Cubical complexes: relative complexes look like:



Theorem 1: H., Solus 2020

Let $\mathcal{C}$ be a shellable polytopal complex with shelling $(F_1, \ldots, F_s)$ and subdivision $\varphi : \mathcal{C}'$. If $(h(_{\sigma(i)}; x))_{i=1}^s$ is an interlacing sequence for some permutation $\sigma \in \mathfrak{S}_s$, then $h(\mathcal{C}'; x)$ is real-rooted.

Application to cubical complexes

- The h-polynomials of barycentric subdivisions of stable relative complexes of a d-cube form an interlacing sequence.
- Same for rth edgewise subdivisions.

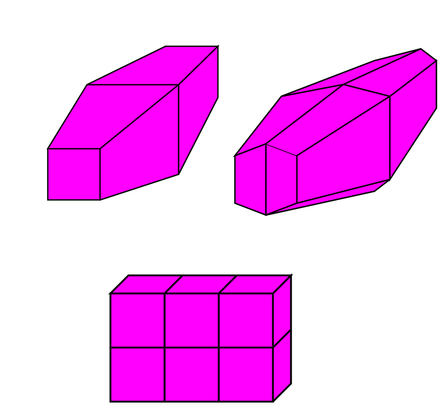
Theorem 1: H., Solus 2020

The barycentric and rth edgewise subdivisions of cubical complexes admitting a stable shelling have real-rooted h-polynomials. This includes the examples to the right.

Families of cubical polytopes

The following cubical complexes have stable shellings.

- boundary complex of capped cubical polytopes
- boundary complex of cuboids
- piles of cubes



Applications to simplicial polytopes

Using this general framework, we can also prove:

- The barycentric subdivision of a shellable simplicial complex has a real-rooted $h-$polynomial (already proven by Brenti and Welker in [2])
- For $r > d$, the rth edgewise subdivision of a shellable simplicial complex has a real-rooted h-polynomial.

Discussion and future work

Since this project was released, Athanasiadis [1] answered Question 1 entirely, generalizing Theorem 2 using different techniques.

Some questions extending the techniques in this project:

- Are their other families of polytopes (not nec. cubical or simplicial) that admit stable shellings, and do real-rootedness results follow?
- Do all polytopes admit a **stable line shelling**?

References

[1] C. A. Athanasiadis. *Face numbers of barycentric subdivisions of cubical complexes.* 2020. arXiv: 2009.02272.

[2] F. Brenti and V. Welker. "f-vectors of barycentric subdivisions". In: *Math. Z.* 259 (2008), pp. 849 –865.

[3] M. Hlavacek and L. Solus. *Subdivisions of shellable complexes.* 2020. arXiv: 2003.07328.

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